Common Fixed Point of Compatible and Weakly Compatible Maps in Fuzzy Metric Spaces

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ABSTRACT

In this paper, we prove a common fixed point theorem for six self mappings in fuzzy metric space using the concept of compatibility and weak compatibility with a functional inequality and only one map is needed to be continuous, which generalizes the result of Singh and Chauhan[1] and Cho[1].

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1. INTRODUCTION.

Zadeh[13] introduced the notion of fuzzy sets in 1965 as a new way to represent vagueness in everyday life. In this paper, we deal with the fuzzy metric space defined by Kramosil and Michalek [8] and modified by George and Veeramani [3]. Jungck [5] introduced the concept of compatible mappings for a pair of self maps. The concept of compatibility in fuzzy metric space was introduced by Mishra et al. [10]. Later on, Jungck [6] generalized the concept of compatibility by introducing the concept of weak compatibility.

Definition 1.1. [13] Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 1.2. [11] A binary operation $* : [0, 1] \times [0, 1] \in [0, 1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a*1 = a$, for all $a \in [0, 1]$,
(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norms are
$a * b = \min\{a, b\}$ (minimum t-norm) and $a * b = a\,b$ (product t-norm).

Definition 1.3. [3] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous, for all x, y, z $\in$ X and t, s > 0.

Let (X, d) be a metric space and let $a*b = a\,b$ or $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$.Then (X, M, *) is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [3].

Definition 1.4. [4] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t >0. Further, the sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all t > 0 and p > 0. The space (X, M, *) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 1.1. [4] Let (X, M, *) be a fuzzy metric space. Then M(x, y, .) is non-decreasing for all $x, y \in X$.

Lemma 1.2. [9] Let (X, M, *) be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.

Throughout this paper (X, M, *) will denote the fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Lemma 1.3.[10] If there exists $k\in(0,1)$ such that $M(x, y, k t) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Lemma 1.4. [7] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0,1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in (0, 1)$.

Lemma 1.5. [2] Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with condition (FM-6) . If there exists a number $k \in (0, 1)$, such that $M(y_{n+2}, y_{n+1}, k\,t) \geq M(y_{n+1}, y_n, t)$ for all t > 0,then $\{y_n\}$ is a Cauchy sequence in X.

Definition 1.5. [10] Let A and B be self mappings on a fuzzy metric space (X, M, *). The pair (A, B) is said to be compatible if $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some $x \in X$.

Definition 1.6. [6] Let A and B be self mappings on a fuzzy metric space (X, M, *). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, Ax = Bx implies ABx = BAx for some x in X.

It is known that a pair (A, B) of compatible maps is weakly compatible but converse is not true in general.

2. MAIN RESULT.

Our result generalizes the results of Singh and Chauhan [12] and Cho[1] in the sense that condition of compatibility of the second pair is replaced by weak compatibility which is lighter condition than that of compatibility, also only one mapping of the first pair is needed to be continuous. We are using another functional inequality and six self maps in our result.

Theorem 2.1. Let (X, M, *) be a complete fuzzy metric space with $r * r \geq r$ for all $r \in [0,1]$ and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :

(2.1.1) $A(X) \subset ST(X)$, $B(X) \subset PQ(X)$
(2.1.2) either A or PQ is continuous;
(2.1.3) (A, PQ) is compatible and (B, ST) is weakly compatible;
(2.1.4) PQ = QP, ST = TS, AQ = QA and BT = TB;
(2.1.5) there exists $q \in (0, 1)$ such that for every x, y in X and t > 0, $M(Ax, By, qt) \geq M(Ax, STy, t) * M(Ax, PQx, t) * M(By, STy, t) * M(PQx, STy, t) * M(PQx, By, 2t)$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let x_0 be an arbitary point in X. As $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$, then there exists $x_1, x_2 \in X$ such that $Ax_0 = STx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$. We can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = STx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Bx_{2n+1} = PQx_{2n+2}$ for n = 0, 1, 2.….

Now, we first show that $\{y_n\}$ is a Cauchy sequence in X.

From (2.1.5), we have

$$M(y_{2n}, y_{2n+1}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt)$$
$$\geq M(Ax_{2n}, STx_{2n+1}, t) * M(Ax_{2n}, PQx_{2n}, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQx_{2n}, STx_{2n+1}, t) * M(PQx_{2n}, Bx_{2n+1}, 2t).$$
$$= M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n-1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, 2t).$$

Using definition 1.2 and definition 1.3, we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t) \quad \text{(i)}$$

Thus we have

$$M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t/q) * M(y_{2n+1}, y_{2n}, t/q) \quad \text{(ii)}$$

Putting (ii) in (i), we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t/q) * M(y_{2n+1}, y_{2n}, t/q)$$

Using lemma 1.1 and lemma 1.4, we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t/q)$$

Proceeding in the similar manner, we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t/q^m)$$

Letting $m\to\infty$ and using (FM-6), we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) \ \forall\, t > 0.$$

In general $M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \ \forall\, t > 0$.

Therefore

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \geq \ldots. \geq M(y_0, y_1, t/q^n)$$

Using (FM-6), we get $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1 \ \forall\, t > 0$.

Now for any positive integer p,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots.. * M(y_{n+p-1}, y_{n+p}, t/p).$$

Therefore $\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1 = 1$.

Thus, $\{y_n\}$ is a Cauchy sequence in X. By completeness of (X, M, *), $\{y_n\}$ converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{PQx_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z in X.

Case I. Suppose A is continuous, we have $A^2x_{2n} \to Az$ and $A(PQ)x_{2n} \to Az$.

The compatibility of the pair (A, PQ) gives that

$\lim_{n\to\infty}(PQ)Ax_{2n} = \lim_{n\to\infty}A(PQ)x_{2n} = Az$.

Step 1. Putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (2.1.5), we have

$M(AAx_{2n}, Bx_{2n+1}, qt) \geq M(AAx_{2n}, STx_{2n+1}, t) * M(AAx_{2n}, PQAx_{2n}, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQAx_{2n}, STx_{2n+1}, t) * M(PQAx_{2n}, Bx_{2n+1}, 2t)$.

Letting $n\to\infty$ and using above results, we get $M(Az, z, qt) \geq M(Az, z, t)$.
Now by Lemma 1.3, we get $Az = z$.

Step 2. Since $A(X) \subset ST(X)$, there exists $u\in X$ such that $z = Az = STu$.
Putting $x = x_{2n}$ and $y = u$ in (2.1.5), we get

$M(Ax_{2n}, Bu, qt) \geq M(Ax_{2n}, STu, t) * M(Ax_{2n}, PQx_{2n}, t) * M(Bu, STu, t) * M(PQx_{2n}, STu, t) * M(PQx_{2n}, Bu, 2t)$.

Letting $n\to\infty$ and using above results, we get $M(z, Bu, qt) \geq M(Bu, z, t)$.
Using Lemma 1.3, we get $z = Bu = STu$. Which implies that u is a coincidence point. The weak compatibility of the pair (B, ST) gives that $STBu = BSTu$ implies that $STz = Bz$.

Step 3. Putting $x = x_{2n}$ and $y = z$ in (2.1.5) , letting $n \to \infty$ and using above results, we get $M(z, Bz, qt) \geq M(z, Bz, t)$ .Using Lemma 1.3, we get $Bz = z$. Thus $STz = Bz = z$.

Step 4. Putting $x = x_{2n}$ and $y = Tz$ in (2.1.5), we get

$M(Ax_{2n}, BTz, qt) \geq M(Ax_{2n}, STTz, t) * M(Ax_{2n}, PQx_{2n}, t) * M(BTz, STTz, t) * M(PQx_{2n}, STTz, t) * M(PQx_{2n}, BTz, 2t)$.

Since $BT = TB$ and $ST = TS$, we have
$BTz = TBz = Tz$ and $ST(Tz) = TS(Tz) = T(STz) = Tz$
Letting $n \to \infty$ and using above results, we get $M(z,Tz, qt) \geq M(z, Tz, t)$.
By using Lemma 1.3, we get $Tz = z$. Now $STz = z$ implies that $Sz = z$.
Hence $Sz = Tz = Bz = z$.

Step 5. As $B(X) \subset PQ(X)$, there exists $v\in X$ such that $z = Bz = PQv$.
Putting $x = v$ and $y = x_{2n+1}$ in (2.1.5), we get

$M(Av, Bx_{2n+1}, qt) \geq M(Av, STx_{2n+1}, t) * M(Av, PQv, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQv, STx_{2n+1}, t) * M(PQv, Bx_{2n+1}, 2t)$.

Taking limit $n \to \infty$ and using above results, we get $M(Av, z, qt) \geq M(Av, z, t)$.

By using Lemma 1.3, we get $Av = z = PQv$. Which implies that v is a coincidence point of (A, PQ). As the pair (A, PQ) is compatible implies weakly compatible.

Therefore $APQv = PQAv$ implies that $Az = PQz$. Hence $PQz = Az = z$.

Step 6. Putting $x = Qz$ and $y = z$ in (2.1.5), we get
$M(AQz, Bz, qt) \geq M(AQz, STz, t) * M(AQz, PQQz, t) * M(Bz, STz, t) * M(PQQz, STz, t) * M(PQQz, Bz, 2t)$.
As $AQ = QA$ and $PQ = QP$, We have
$AQz = QAz = Qz$ and $PQ(Qz) = QP(Qz) = Q(PQz) = Qz$.

Using above results, we get $M(Qz, z, qt) \geq M(Qz, z, t)$.

By using Lemma 1.3, $Qz = z$. Therefore, $PQz = z$ implies that $Pz = z$.

Hence $Az = Bz = Sz = Tz = Pz = Qz = z$
Thus, z is a common fixed point of A, B, S, T, P and Q.

Case II. Suppose PQ is continuous, we have
$PQPQx_{2n} \to PQz$ and $(PQ)Ax_{2n} \to PQz$.
As the pair (A, PQ) is compatible, we have
$\lim_{n\to\infty}A(PQ)x_{2n} = \lim_{n\to\infty}(PQ)Ax_{2n} = PQz$.
It can be proved easily that z is a common fixed point of A, B, S, T, P and Q.

Uniqueness.
Let w be another common fixed point of A, B, S, T, P and Q, then

$$w = Aw = Bw = Sw = Tw = Pw = Qw.$$

Putting $x = z$ and $y = w$ in (2.1.5), we get $M(z, w, qt) \geq M(z, w, t)$.
Now by Lemma 1.3, we get $z = w$.
Therefore, z is unique common fixed point of A, B, S, T, P and Q.

Remark 2.1. If we take $Q = T = I$ in theorem 2.1 then the condition (2.1.4) is satisfied trivially and we get the following result.

Corollary 2.1. Let (X, M, *) be a complete fuzzy metric space with $r*r \geq r$ for all $r\in [0,1]$ and let A, B, S and P be mappings from X into itself such that the following conditions are satisfied :
(2.1.6) $A(X) \subset S(X)$, $B(X) \subset P(X)$
(2.1.7) either A or P is continuous;
(2.1.8) (A, P) is compatible and (B, S) is weakly compatible;
(2.1.9) there exists $q \in (0, 1)$ such that for every x, y in X and $t > 0$,
$M(Ax, By, qt) \geq M(Ax, Sy, t) * M(Ax, Px, t) * M(By, Sy, t) * M(Px, Sy, t) * M(Px, By, 2t)$.

Then A, B, S and P have a unique common fixed point in X.

Remark 2.2. If we take a * b = min{a, b} where a, b∈ (0,1) then in view of remark 2.1, corollary 2.1 is a generalization of the result of Singh and Chauhan[12], as only one mapping of the first pair is needed to be continuous and second pair of mappings is weakly compatible in (2.1.8).

Remark 2.3. In view of remark 2.1, corollary 2.1 is also a generalization of the result of Cho[1] in the sense of another functional inequality (2.1.9), weak compatibility of second pair and continuity for only one mapping in the first pair of (2.1.8).

Corollary 2.3. Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and the pairs (A, PQ) and (B, ST) are compatible.
Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. As compatibility implies weak compatibility, the proof follows from theorem 2.1.

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